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Arnold D. Kerr[1] and Warren T. Palmer[2]

SUMMARY

In the past, the analyses of floating ice plates subjected to static or dynamic loads were based on the theory of a thin homogeneous plate, although in actual floating ice plates Young's modulus may vary strongly with depth. Recently, A. Assur concluded, on the basis of a heuristic argument, that the solutions obtained for homogeneous plates may be used for floating ice plates, if a modified flexural rigidity is used. The purpose of the present paper is to study this question, by establishing a mathematically consistent formulation for the dynamic plate equation utilizing Hamilton's Principle in conjunction with the three dimensional theory of elasticity. It was found that for a variable Young's modulus and a constant Poisson's ratio the resulting formulations for plates and beams are the same as those for the corresponding homogeneous problems, if a modified flexural rigidity is used; thus confirming Assur's conclusion. It is shown that the corresponding stress distribution is not linear and that the formula $\sigma = Mz_o/I$ used by a number of investigators for the determination of failure stresses from tests on floating ice beams, is not applicable. A correct formula is derived and its use discussed.

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INTRODUCTION

The first analysis of a floating ice plate is due to H. Hertz [1]. This analysis, published in 1884, deals with an infinite plate subjected to a lateral concentrated force, and is based on the differential equation of the homogeneous elastic plate

$$D \nabla^4 w + kw = q \tag{1}$$

In above equation, w(x,y) is the deflection, q(x,y) is the lateral load, D is the flexural rigidity, k is the specific weight of the liquid, and

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

A large number of solutions of equ. (1) for circular plates are presented in a book by F. Schleicher [2] published in 1926. S. Bernstein [3], in 1929, discussed the stresses and deformations of a floating ice plate, by comparing analytical results based on equ. (1) with test data. He found that for loads of <u>short duration</u> the ice plate response was elastic. A number of problems solved by H.M. Westergaard in connection with stresses in concrete pavements are also based on equ. (1). These and related problems are discussed in the book by S. Timoshenko and S. Woinowsky-Krieger [4]. Additional solutions of equ. (1) are contained in References [5-8]. Very recent discussions of floating ice plates by H. Brunk [9] and O. Mahrenholtz [10] are also based on equ. (1).

A dynamic version of equ. (1) is (see Fig. 1)

$$D \nabla^4 w + p + m \frac{\partial^2 w}{\partial t^2} = q \tag{2}$$

where p(x,y,t) is the dynamic pressure which the liquid base exerts upon the bottom surface of the floating plate, m is the mass of the plate per unit area, and q(x,y,t) represents the static intensity as well as the inertia of the load. Equ. (2) was used by a number of investigators to analyze the response of floating ice plates subjected to dynamic loads. A large number of these results are presented and discussed in the books by W.M. Ewing, W.S. Jardetzky and F. Press [11] and by D.E. Kheishin [12]. Very recent investigations of floating ice plates, which are also based on equ. (2), were published by D.E. Nevel [13], [14], H. Reissman [15], and P.F. Sabodash and I. G. Filippov [16].

In an actual floating ice plate the temperature varies throughout the thickness. Namely, it is about +32°F (freezing temperature) at the bottom surface, which rests on the water, and usually much lower at the upper surface, which is in contact with the outside air. Because of this temperature gradient and the sensitivity of ice properties near the freezing temperature, Young's modulus E varies substantially throughout the plate thickness. The corresponding variation of Poisson's ratio ν appears to be very small [17]. In 1966, A. Assur [18], discussing this problem, proposed a correction of the classical plate equation (1), assuming that ν is constant but E varies with depth. Using a heuristic argument, Assur concluded that the solutions of (1) remain the same, except that $D = Eh^3/[12(1-\nu^2)]$ has to be replaced by a different value. Identical results were obtained, independently, by M. Newman and M. Forray [19] while discussing the effect of aerodynamic heating on elastic plates. Neither of these papers discussed the boundary conditions to be used, which affect the solution of a particular problem.

In view of the novelty of the assumption that in the plate E varies with depth, there is a need, at first, to establish a mathematically consistent formulation. Namely, to establish the proper differential equation and the corresponding boundary conditions for the analysis of floating ice plates which respond elastically. This is also necessary in order to determine if all solutions obtained for the homogeneous plate are applicable to floating ice plate problems just by changing the flexural rigidity of the plate. This and related topics are studied in the following.

THE FORMULATION OF THE FLOATING ICE PLATE WITH E(z)

Consider a floating ice plate subjected to a lateral load $q(x,y,t)$ as shown in Fig. 1. We denote by $p(x,y,t)$ the pressure the liquid exerts upon the plate due to q.

In order to insure a mathematically consistent formulation, we utilize Hamilton's Principle for elastic bodies [20]

$$\delta \int_{t_o}^{t_1} (U - W - K)dt = 0 \tag{3}$$

where U is the elastic strain energy stored in the plate, W is work potential of the outside forces q and p, and K is the kinetic energy of the plate.

In the following we analyze the problem within the frame work of the linear bending theory of plates.

<u>The elastic strain energy of the plate</u> is determined from the general expression

$$U = \frac{1}{2} \iiint_V \left(\sigma_{xx}\epsilon_{xx} + \sigma_{yy}\epsilon_{yy} + \sigma_{zz}\epsilon_{zz} + \sigma_{xy}\epsilon_{xy} + \sigma_{xz}\epsilon_{xz} + \sigma_{yz}\epsilon_{yz}\right)dV \tag{4}$$

where the integration extends over the undeformed volume of the plate, V. Denoting by (u,v,w) the components of the displacement vector of any point (x,y,z) of the plate, it follows that

$$\begin{aligned}
\epsilon_{xx} &= u,_x & \epsilon_{xy} &= \epsilon_{yx} = u,_y + v,_x \\
\epsilon_{yy} &= v,_y & \epsilon_{xz} &= \epsilon_{zx} = u,_z + w,_x \\
\epsilon_{zz} &= w,_z & \epsilon_{yz} &= \epsilon_{zy} = v,_z + w,_y
\end{aligned} \tag{5}$$

where $(\ ),_x = \partial(\ )/\partial x$, etc.

For thin plates in bending it is reasonable to assume that

$$\sigma_{zz} \ll \sigma_{xx} \quad ; \quad \sigma_{zz} \ll \sigma_{yy} \tag{6}$$

Hence σ_{zz} is negligible compared to σ_{xx} or σ_{yy}.

Because the plate under consideration is relatively thin and the expected strains relatively small, it is assumed that the displacements w of each point on a line parallel to the z-axis, are approximately the same. Hence

$$w(x,y,z) \cong \hat{w}(x,y) \tag{7}$$

Therefore, to describe the vertical displacements of the plate, it is sufficient to use only one plane parallel to the x,y-plane as a "reference" plane. We choose, the x,y-plane as this plane and place it so that it coalesces with the "neutral" plane (at which σ_{xx} and σ_{yy} due to bending are zero), as shown in Fig. 2. Assumption (7) implies

$$\epsilon_{zz} \cong 0 \tag{8}$$

throughout the plate.

It should be noted that even if the plane section hypothesis is adopted, the stress distribution is <u>not</u> linear in the z direction because $E = E(z)$ and therefore the position of the neutral plane does not, in general, coincide with the middle plane. Hence for the problem under consideration the position of the x,y-reference plane is, at first, not known. Its location depends upon $E(z)$. Its determination is discussed later.

The usual assumption that a line normal to the x,y-plane remains straight after deformation, is expressed as

$$\begin{aligned} u(x,y,z) &= \hat{u}(x,y) + z\,\varphi(x,y) \\ v(x,y,z) &= \hat{v}(x,y) + z\,\psi(x,y) \end{aligned} \tag{9}$$

where $\hat{u}$ and $\hat{v}$ are the respective displacement components of the reference plane. The additional assumption that this straight line remains perpendicular to the deformed reference plane implies

$$\epsilon_{xz} = 0 \qquad \epsilon_{yz} = 0 \tag{10}$$

With (6), (8), and (10), the strain energy expression (4) reduces to

$$U = \frac{1}{2} \iiint\limits_V \left(\sigma_{xx}\epsilon_{xx} + \sigma_{yy}\epsilon_{yy} + \sigma_{xy}\epsilon_{xy}\right)dV \tag{11}$$

The functions φ and ψ are determined by substituting (9) into (10), noting (5). The obtained expressions are

$$\varphi(x,y) = -\hat{w}_{,x} \qquad \psi(x,y) = -w_{,y} \tag{12}$$

Hence, according to (9)

$$\begin{aligned} u(x,y,z) &= \hat{u}(x,y) - z\,\hat{w}_{,x} \\ v(x,y,z) &= \hat{v}(x,y) - z\,\hat{w}_{,y} \\ w(x,y,z) &= \hat{w}(x,y) \end{aligned} \tag{13}$$

Noting that in the classical bending theory $\hat{u} \equiv 0$ and $\hat{v} \equiv 0$, it follows from (5) that

$$\begin{aligned} \epsilon_{xx} &= -z\,\hat{w}_{,xx} \\ \epsilon_{yy} &= -z\,\hat{w}_{,yy} \\ \epsilon_{xy} &= -2z\,\hat{w}_{,xy} \end{aligned} \tag{14}$$

Because of the assumption that σ_{zz} is negligibly small compared to σ_{xx} or σ_{yy}, Hooke's law becomes

$$\begin{aligned} \sigma_{xx} &= \frac{E}{1-\nu^2}\left(\epsilon_{xx} + \nu\,\epsilon_{yy}\right) \\ \sigma_{yy} &= \frac{E}{1-\nu^2}\left(\epsilon_{yy} + \nu\,\epsilon_{xx}\right) \\ \sigma_{xy} &= \frac{E}{2(1+\nu)}\,\epsilon_{xy} \end{aligned} \tag{15}$$

Substituting (15) into (11) we obtain

$$U = \frac{1}{2} \iiint\limits_V \left(\frac{E}{1-\nu^2} \epsilon^2_{xx} + 2 \frac{E\nu}{1-\nu^2} \epsilon_{xx}\epsilon_{yy} + \frac{E}{1-\nu^2} \epsilon^2_{yy} + \frac{E}{2(1+\nu)} \epsilon^2_{xy} \right) dV \tag{16}$$

Because $E = E(z)$ equ. (16) may be rewritten, noting (14), as follows

$$U = \frac{1}{2} \iint\limits_R \left[D_1 \left(\hat{w},_{xx} + \hat{w},_{yy} \right)^2 + 2D_2 \left(\hat{w},^2_{xy} - \hat{w},_{xx} \hat{w},_{yy} \right) \right] dxdy \tag{17}$$

where the integration extends over the undeformed region of the plate, R, as shown in Fig. 3 and

$$\begin{aligned} D_1 &= \int_{-z_o}^{h-z_o} \frac{z^2 E(z)}{1-\nu^2} dz = \frac{1}{1-\nu^2} \int_{-z_o}^{h-z_o} z^2 E(z) dz \\ D_2 &= \int_{-z_o}^{h-z_o} \frac{z^2 E(z)}{1+\nu} dz = (1-\nu) D_1 \end{aligned} \tag{18}$$

The work potential of the outside forces is written down, as usually done in the classical plate theory, by assuming that the horizontal components of the pressure p are negligibly small. Hence

$$W = \iint\limits_R (q-p)\, \hat{w}\, dA - \int\limits_B M^o_n\, \hat{w},_n ds + \int\limits_B V^o_n\, \hat{w}\, ds \tag{19}$$

where $M^o_n(s)$ is a prescribed bending moment (per unit length of plate boundary) acting along the boundary in the positive sense and $V^o_n(s)$ is a prescribed vertical line force (per unit length of boundary) acting along the boundary in the direction of the z-axis.

The kinetic energy of the plate is written down under the assumption that the load q generates waves of large wave length [15], and therefore the rotational kinetic energy is negligible compared to the translational. Hence

$$K = \frac{1}{2} \iint\limits_R m\, \hat{w},^2_t \, dA \tag{20}$$

where m = gρh is the mass of the plate per unit area.

Substituting expressions (17), (19) and (20) into Hamilton's Principle (3), and performing the indicated variations, we obtain noting that $\delta w = 0$ at $t = t_1$ and $t = t_2$ (for details see [5] or [21]).

$$\delta \int_{t_o}^{t_1} (U-W-K)dt =$$

$$= \int_{t_o}^{t_1} \Big\{ \iint_R \Big[(D_1 \nabla^2 \hat{w}),_{xx} + (D_1 \nabla^2 \hat{w}),_{yy} + m\, \hat{w},_{tt} - (q-p) \Big] \delta\hat{w}\, dA -$$

$$- \int_B \Big[(D_1 \nabla^2 \hat{w}),_n - D_2 [(\hat{w},_{xx} - \hat{w},_{yy}) \sin\alpha \cos\alpha - \hat{w},_{xy} \cos 2\alpha],_s + V_n^o \Big] \delta\hat{w}\, ds$$

$$+ \int_B \Big[D_1 \nabla^2 \hat{w} - D_2 (\hat{w},_{xx} \sin^2\alpha + \hat{w},_{yy} \cos^2\alpha - \hat{w},_{xy} \sin 2\alpha) + M_n^o \Big] \delta\hat{w},_n ds \Big\} dt = 0 \qquad (21)$$

When the boundary deflections $\hat{w}(s,t)$ and the boundary rotations $\hat{w},_n(s,t)$ are prescribed, then the boundary integrals in (21) vanish. In this case, equ. (21) is satisfied, according to the fundamental lemma, when

$$(D_1 \nabla^2 \hat{w}),_{xx} + (D_1 \nabla^2 \hat{w}),_{yy} + m\, \hat{w},_{tt} = q - p \qquad \text{in } R \qquad (22)$$

For h = const, and because E = E(z), the flexural rigidity D_1 does not depend upon x and y and equ. (22) may be written as follows:

$$D_1 \left(\frac{\partial^4 \hat{w}}{\partial x^4} + 2 \frac{\partial^4 \hat{w}}{\partial x^2 \partial y^2} + \frac{\partial^4 \hat{w}}{\partial y^4} \right) + m \frac{\partial^2 \hat{w}}{\partial t^2} = q - p \qquad \text{in } R \qquad (23)$$

It should be noted that when $\hat{w}$ is time independent, hence $\hat{w} = \hat{w}(x,y)$, then $\hat{w},_{tt} \equiv 0$ and

$$p(x,y) = k\hat{w}(x,y) \qquad (24)$$

where k is the specific weight of the liquid. When w = w(x,y,t), the pressure p(x,y,t) is determined from the equations of fluid dynamics. For details of this procedure the reader is referred to Ref. [12].

From the above discussion it follows that when the ice plate with $E(z)$ is "clamped" along the entire boundary, the formulation for the plate consists of the differential equation (23), the boundary conditions

$$\left.\begin{aligned} \hat{w}(s,t) &= 0 \\ \hat{w},_n(s,t) &= 0 \end{aligned}\right\} \quad \text{on } B \tag{25}$$

and the two initial conditions

$$\begin{aligned} \hat{w}(x,y,o) &= f(x,y) \\ \hat{w},_t(x,y,o) &= g(x,y) \end{aligned} \tag{26}$$

Noting that differential equation (23) is, except for the coefficient D_1, identical with equ. (2) and that boundary conditions (25) and initial conditions (26) are the same as those of a homogeneous plate, it follows that all solutions obtained for the clamped homogeneous plate may be used for floating ice plates with $E(z)$, by replacing $D = Eh^3/[12(1-\nu^2)]$ with

$$D_1 = \frac{1}{1-\nu^2}\int_{-z_o}^{h-z_o} z^2 E(z)dz$$

It is obvious that this is also the case for the infinite plate.

When the plate boundary is "free", the two boundary conditions which satisfy (21) are, noting (18),

$$\begin{aligned} D_1\left[-\nabla^2\hat{w} + (1-\nu)(\hat{w},_{xx}\sin^2\alpha + \hat{w},_{yy}\cos^2\alpha - \hat{w},_{xy}\sin 2\alpha)\right]_{(s,t)} &= M_n^o(s,t) \\ D_1\left[-(\nabla^2\hat{w}),_n + (1-\nu)\,[(\hat{w},_{xx}-\hat{w},_{yy})\sin\alpha\cos\alpha - \hat{w},_{xy}\cos 2\alpha],_s\right]_{(s,t)} &= V_n^o(s,t) \end{aligned} \tag{27}$$

where $M_n^o(s,t)$ and $V_n^o(s,t)$ are prescribed functions. A comparison reveals, that except for the coefficient D_1, boundary conditions (27) are identical with those of the homogeneous plate.

From the above discussion it follows that all solutions $\hat{w}$ obtained in the literature for the homogeneous plate using equations (1) or (2), are also valid for the corresponding ice plate with $E = E(z)$ and ν = const. when D is replaced by D_1.

DETERMINATION OF STRESSES

Substituting (14) into (15), we obtain

$$\begin{aligned} \sigma_{xx} &= - \frac{z\,E(z)}{1-\nu^2} (\hat{w},_{xx} + \nu\,\hat{w},_{yy}) \\ \sigma_{yy} &= - \frac{z\,E(z)}{1-\nu^2} (\hat{w},_{yy} + \nu\,\hat{w},_{xx}) \\ \sigma_{xy} &= - \frac{z\,E(z)}{1+\nu} \hat{w},_{xy} \end{aligned} \tag{28}$$

It follows that, although the plane section hypothesis was adapted, because $E = E(z)$ the distribution of the stresses σ_{xx}, σ_{yy}, and σ_{xy} is <u>not linear</u> in z.

Once $E(z)$ and ν are given and $\hat{w}(x,y)$ is obtained from the formulation discussed before, the stresses throughout the plate may be determined from (28).

In order to correlate moments and stresses, we note that

$$M_x = \int_{-z_o}^{h-z_o} \sigma_{xx} z dz \quad ; \quad M_y = \int_{-z_o}^{h-z_o} \sigma_{yy} z dz \quad ; \quad M_{xy} = \int_{-z_o}^{h-z_o} \sigma_{xy} z dz \tag{29}$$

In view of the relations in (28) above equations become

$$\begin{aligned} M_x &= - D_1 (\hat{w},_{xx} + \nu\,\hat{w},_{yy}) \\ M_y &= - D_1 (\hat{w},_{yy} + \nu\,\hat{w},_{xx}) \\ M_{xy} &= - (1-\nu)\, D_1\, \hat{w},_{xy} \end{aligned} \tag{30}$$

From (28) and (30) it follows that

$$\sigma_{xx} = \frac{M_x}{(1-\nu^2)D_1} z\, E(z)$$

$$\sigma_{yy} = \frac{M_y}{(1-\nu^2)D_1} z\, E(z) \qquad (31)$$

$$\sigma_{xy} = \frac{M_{xy}}{(1-\nu^2)D_1} z\, E(z)$$

Note that only when E = const., do the equations in (31) reduce to the usual relations for a homogeneous plate

$$\sigma_{xx} = \frac{M_x z}{h^3/12} \quad ; \quad \sigma_{yy} = \frac{M_y z}{h^3/12} \quad ; \quad \sigma_{xy} = \frac{M_{xy} z}{h^3/12} \qquad (32)$$

DETERMINATION OF D_1

As shown in (18), the flexural rigidity of the floating ice plate with $E = E(z)$ and ν = const., is

$$D_1 = \frac{1}{1-\nu^2} \int_{-z_o}^{h-z_o} z^2 E(z)dz$$

where z_o is, as yet, an unknown quantity. Its determination is discussed in the following.

In formulating the ice plate problem it was assumed that the used reference plane coalesces with the neutral plane, that is with the plane at which σ_{xx} and σ_{yy} are zero throughout the plate. This and the additional assumption that $\hat{u}(x,y)$ and $\hat{v}(x,y)$ are zero imply that the resultant normal forces are zero. Hence the equations

$$\int_{-z_o}^{h-z_o} \sigma_{xx}dz = 0 \qquad \text{and} \qquad \int_{-z_o}^{h-z_o} \sigma_{yy}dz = 0 \qquad (33)$$

should be satisfied throughout the plate. Substituting σ_{xx} and σ_{yy} from (28) into the above conditions, we obtain

$$\int_{-z_o}^{h-z_o} \frac{z\,E(z)}{1-\nu^2}(\hat{w}_{,xx} + \nu\,\hat{w}_{,yy})dz = 0$$

$$\int_{-z_o}^{h-z_o} \frac{z\,E(z)}{1-\nu^2}(\hat{w}_{,yy} + \nu\,\hat{w}_{,xx})dz = 0 \tag{34}$$

Because $\hat{w}_{,xx}$ and $\hat{w}_{,yy}$ do not depend upon z, and ν is a constant, both equations in (34) are satisfied when

$$\int_{-z_o}^{h-z_o} z\,E(z)dz = 0 \tag{35}$$

This is the equation for the determination of z_o; its only unknown.

Thus for a given E(z), the value of z_o is determined from condition (35) and then the flexural rigidity of the ice plate, D_1, using the first relation in (18). With D_1 known, the formulation of an ice plate problem with E(z) as discussed above, is completed.

To demonstrate the use of above relations, let us assume, that the distribution of E can be expressed analytically as follows:

$$E = E_o\left[1 - (1-\alpha)\left(\frac{z}{h} + \frac{z_o}{h}\right)^n\right] \tag{36}$$

where $0 \leq \alpha < 1$ and n are determined from a curve fitting analysis. Graphical representations of expression (36) for $\alpha = 0.2$ and n = 0.5,1.0, and 2.0, are shown in Fig. 4.

Substitution of (36) into equ. (35), yields

$$z_o = h\,\frac{(n+2\alpha)(n+1)}{2(n+2)(n+\alpha)} \tag{37}$$

Substitution of (36) into the first equation in (18), yields

$$D_1 = \frac{E_o h^3}{1-\nu^2}\left[\frac{n+3\alpha}{3(n+3)} - \frac{n+2\alpha}{n+2}\left(\frac{z_o}{h}\right) + \frac{n+\alpha}{n+1}\left(\frac{z_o}{h}\right)^2\right] \tag{38}$$

Because of (37), D_1 becomes

$$D_1 = \frac{E_o h^3}{12(1-\nu^2)}\left[\frac{4(n+2)^2(n+\alpha)(n+3\alpha) - 3(n+1)(n+3)(n+2\alpha)^2}{(n+2)^2(n+3)(n+\alpha)}\right] \tag{39}$$

The first term in (39) is the flexural rigidity of a homogeneous plate with a constant Young's modulus E_o. Hence the term in the brackets is the reduction coefficient due to the variation of E. For example, for $\alpha = 0.2$ and $n = 2$ (shown in Fig. 4)

$$D_1 = 0.61 \frac{E_o h^3}{12(1-\nu^2)} \tag{40}$$

and for $\alpha = 0.2$ and $n = 0.5$

$$D_1 = 0.42 \frac{E_o h^3}{12(1-\nu^2)} \tag{41}$$

The corresponding bending stress distributions are shown in Fig. 5. It should be noted that the largest stresses under the neutral plane do not necessarily occur in the bottom fibers.

Note that D_1 may also be determined by subjecting a floating ice plate to a load, by recording at some points the instantaneous deflections or slopes, and then by comparing them with the corresponding values based on equ. (23).

THE FLOATING ICE BEAM AND THE STRENGTH TEST

To determine the strength of floating ice plates, a number of investigators used as a test sample a floating cantilever beam, cut out from a floating ice sheet, which is loaded at the free end until failure, as shown in Fig. 6. The failure stress was calculated using the stress formula for the largest stress of a homogeneous beam (References [22-26])

$$\sigma_{xx} = \frac{Mz_o}{I} \tag{42}$$

In this expression M is the bending moment at the broken cross section at the instant of failure, z_o is the distance of neutral axis from upper or lower surface of tested beam, and I is the moment of inertia with respect to a horizontal line which passes through and is normal to the beam axis. In view of the variation of E throughout the cross section, the evaluation of the failure stress using (42) is not permissible. The correct equation is derived in the following.

It should be noted that after the beam is cut out, also a large part of the side surfaces of the beam gets in contact with the liquid which is at about +32°F. This in turn affects a variation of E, and hence of σ_{xx}, also in the y direction. An additional complication is created by the fact that after the beam is cut out, the variation of E in the y direction varies with time until a thermal steady state is established in the beam.

This complication can be avoided if the beam is formed by initiating the cutting at the tip and after the root of the beam is formed, to load the beam immediately until it fails. In this case it appears reasonable to assume that $E = E(z)$ only, and hence also the bending stresses σ_{xx} will not vary with y (see Fig. 6).

The formulation of such a beam problem is very similar to the one of the plate discussed above. In view of the usual assumptions

$$\sigma_{yy} \ll \sigma_{xx} \qquad \sigma_{zz} \ll \sigma_{xx} \tag{43}$$

$$\epsilon_{yz} = 0 \quad ; \quad \epsilon_{xy} = 0 \quad ; \tag{44}$$

$$\epsilon_{xz} = 0 \tag{45}$$

it follows that

$$U = \frac{1}{2} \iiint_V \sigma_{xx} \epsilon_{xx} \, dV \tag{46}$$

Because of assumption (45)

$$u = \hat{u}(x) - z\,\varphi(x) = \hat{u}(x) - z\hat{w}_{,x} \tag{47}$$

With $\hat{u} \equiv 0$

$$\epsilon_{xx} = u_{,x} = - z\hat{w}_{,xx} \tag{48}$$

Setting

$$w \cong \hat{w}(x) \tag{49}$$

and noting that Hooke's law reduces, because of (43), to

$$\sigma_{xx} = E \, \epsilon_{xx} \tag{50}$$

we obtain

$$U = \frac{1}{2} \int_{x_1}^{x_2} I_1 \, \hat{w},^2_{xx} \, dx \tag{51}$$

where

$$I_1 = \iint_A z^2 E(z) dA \tag{52}$$

The integration extends over the area of beam cross section A.

From the obtained U, equ. (51), it follows that the differential equation and boundary conditions for a beam with E(z) are the same as for a corresponding homogeneous beam if the flexural rigidity EI is replaced by I_1.

According to (50) and (48), the stress

$$\sigma_{xx} = E\epsilon_{xx} = - zE(z) \, \hat{w},_{xx} \tag{53}$$

Noting that

$$M_x = \iint_A \sigma_{xx} z dA = - \hat{w},_{xx} I_1 \tag{54}$$

it follows that

$$\sigma_{xx} = \frac{M_x}{I_1} z \, E(z) \tag{55}$$

Hence for the determination of failure stresses from tests on floating beams, equ. (55) has to be used instead of equ. (42).

It should be noted that the distribution of σ_{xx}, which correspond to E(z) profiles shown in Fig. 4 are, except for a coefficient, identical to those shown in Fig. 5 since, also for the beam problem, z_o is determined from equ. (35). The fact, that for a given bending moment the stresses at the bottom fibers are smaller than the corresponding stresses at the top fibers [as well as those obtained from equ. (42)] may be the reason for the observation that "the strength of the cantilever beams was greater when the bottom of the ice was put in tension" [25].

ADDITIONAL REMARKS

Recently, O. Mahrenholtz [10] discussing the response of floating ice plates, suggested to include the condition

$$\iint_R \hat{w} dA = 0 \tag{56}$$

as part of the formulation. Condition (56) implies that an incompressible liquid is sealed between the plate and a rigid liquid-tight surrounding. This type of problems were studied for an incompressible liquid in Ref. [7] and for a compressible liquid in Ref. [8]. The obtained results indicate that when analyzing the response of an ice plate which covers a river or lake, there is no justification to impose condition (56).

For those cases when condition (56) has to be imposed, it may be incorporated in the above analysis by means of the Lagrange multiplier method.

The results obtained in the present paper suggest that a consistent formulation has to be also derived for the analysis of viscoelastic deformations of floating ice sheets, which takes into consideration the variation of the material parameters with plate depth. This is necessary, in particular, when stress distributions are studied, as done by J.L. Cutcliffe, W.D. Kingery and R.L. Coble [27]. The above remark also applies to the paper by H.A. Hobbs, J.L. Cutcliffe and W.D. Kingery [28]. In connection with this paper, it should be noted that if the shapes of two deflection surfaces are approximately the same (in the sense of comparing two graphs), then this does <u>not</u> imply that the stresses will also be approximately the same.

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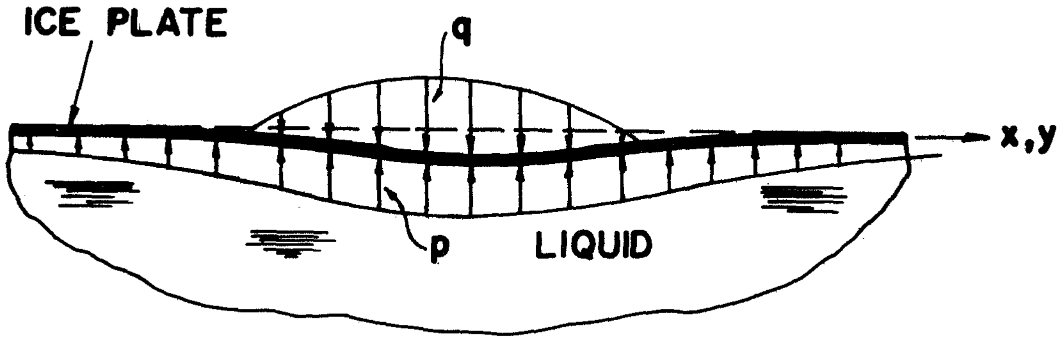

Fig. 1

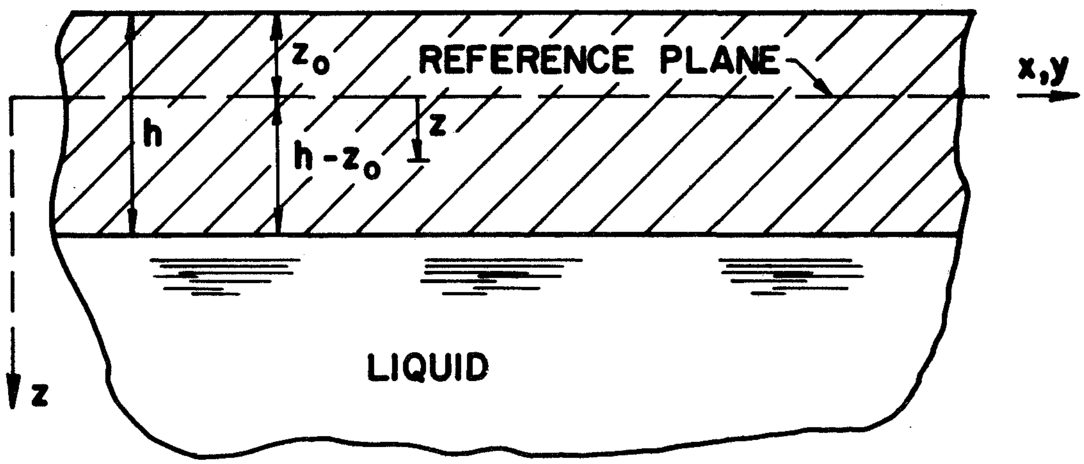

Fig. 2

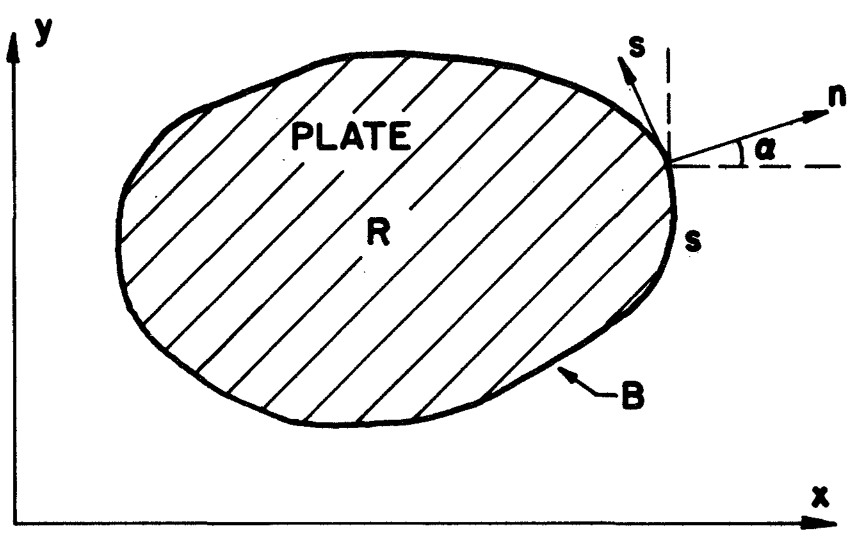

Fig. 3

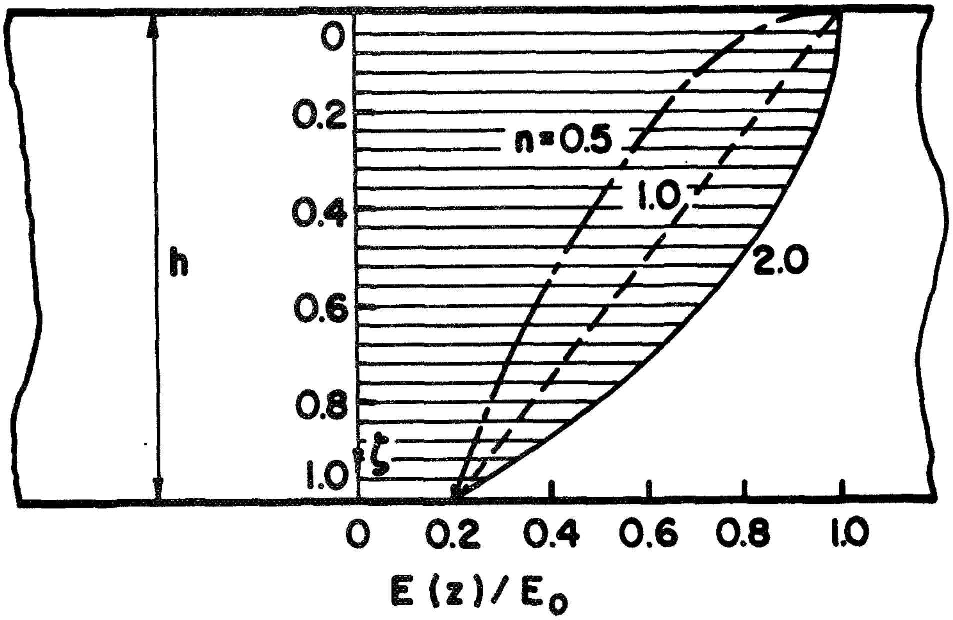

Fig. 4 Distribution of E according to equ. (36)

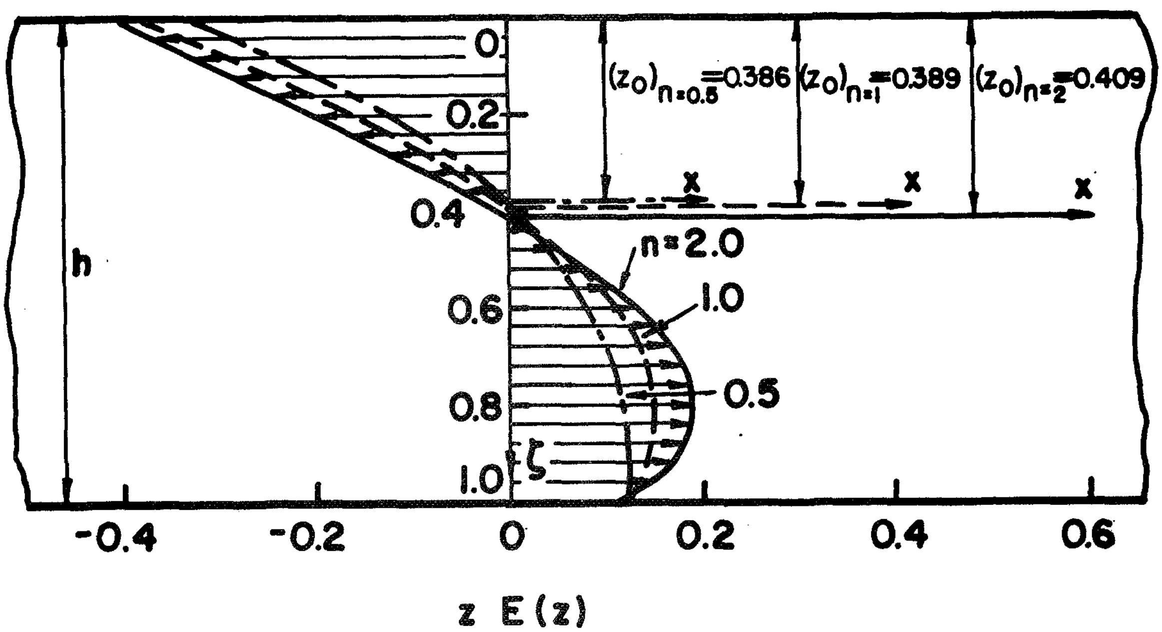

Fig. 5 Distribution of stresses (multiplied by a constant)

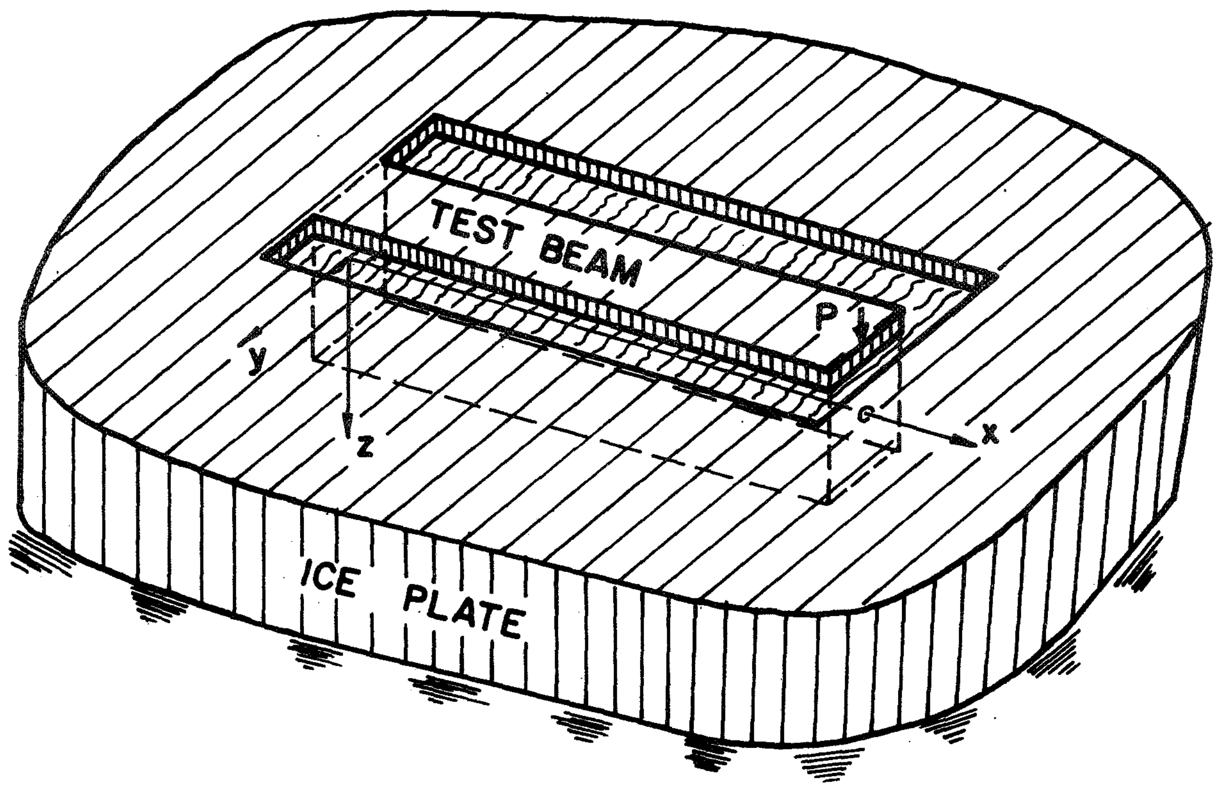

Fig. 6